Mathematische Grundlagen

Matrizenalgebra und Matrizenanalysis

Skalar $\mu \in \mathbb{R}$

Vektoren $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $x_i \in \mathbb{R}$

$\mathbf{y} \in \mathbb{R}^m$, $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$, $y_i \in \mathbb{R}$

Matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{A} = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}$, $A_{ij} \in \mathbb{R}$

Elementare Operationen

Operation	Schreibweise	Komponenten	Abbildung
Addition	$\mathbf{C} = \mathbf{A} + \mathbf{B}$	$C_{ij} = A_{ij} + B_{ij}$	$\mathbb{R}^{m\times n} \times \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$
Multiplikation mit Skalar	$\mathbf{C} = \mu\,\mathbf{A}$	$C_{ij} = \mu\, A_{ij}$	$\mathbb{R} \times \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$
Transponieren	$\mathbf{C} = \mathbf{A}^T$	$C_{ij} = A_{ji}$	$\mathbb{R}^{m\times n} \to \mathbb{R}^{n\times m}$
Differentiation	$\mathbf{C} = \frac{d}{dt}\mathbf{A}$ $\mathbf{C} = \frac{\partial \mathbf{x}}{\partial \mathbf{y}}$	$C_{ij} = \frac{d}{dt}A_{ij}$ $C_{ij} = \frac{\partial x_i}{\partial y_j}$	$\mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times m}$
Matrizenmultiplikation	$\mathbf{y} = \mathbf{A} \cdot \mathbf{x}$ $\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$	$y_i = \sum_k A_{ik}\, x_k$ $C_{ij} = \sum_k A_{ik}\, B_{kj}$	$\mathbb{R}^{m\times n} \times \mathbb{R}^n \to \mathbb{R}^m$ $\mathbb{R}^{m\times n} \times \mathbb{R}^{n\times p} \to \mathbb{R}^{m\times p}$
Inneres Produkt (Skalarprodukt)	$\mu = \mathbf{x} \cdot \mathbf{z}$	$\mu = \sum_k x_k\, z_k$	$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$
Äußeres Produkt (dyadisches Produkt)	$\mathbf{A} = \mathbf{x}\,\mathbf{y}$	$A_{ij} = x_i\, y_j$	$\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times m}$

Rechenregeln

Addition: $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$

$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

Multiplikation mit Skalar: $\mu(\mathbf{A} \cdot \mathbf{B}) = (\mu\mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot (\mu\mathbf{B})$

$\mu(\mathbf{A} + \mathbf{B}) = \mu\mathbf{A} + \mu\mathbf{B}$

Transposition: $(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A}$

$(\mathbf{A} + \mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}$

$(\mu\,\mathbf{A})^{\mathrm{T}} = \mu\,\mathbf{A}^{\mathrm{T}}$

$(\mathbf{A} \cdot \mathbf{B})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}} \cdot \mathbf{A}^{\mathrm{T}}$

Differentiation: $\frac{\mathrm{d}}{\mathrm{dt}}(\mathbf{A} + \mathbf{B}) = \frac{\mathrm{d}}{\mathrm{dt}}\mathbf{A} + \frac{\mathrm{d}}{\mathrm{dt}}\mathbf{B}$

$\frac{\mathrm{d}}{\mathrm{dt}}(\mathbf{A} \cdot \mathbf{B}) = \left(\frac{\mathrm{d}}{\mathrm{dt}}\mathbf{A}\right) \cdot \mathbf{B} + \mathbf{A} \cdot \left(\frac{\mathrm{d}}{\mathrm{dt}}\mathbf{B}\right)$

$\frac{\mathrm{d}}{\mathrm{dt}}\mathbf{x}(\mathbf{y}) = \frac{\partial \mathbf{x}}{\partial \mathbf{y}} \cdot \frac{\mathrm{dy}}{\mathrm{dt}}$

Matrizenmultiplikation: $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$

$\mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C}) = (\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C}$

aber i.A. $\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$

Skalarprodukt: $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$

$\mathbf{x} \cdot \mathbf{x} \geq 0 \quad \forall\mathbf{x}, \qquad \mathbf{x} \cdot \mathbf{x} = 0 \;\Leftrightarrow\; \mathbf{x} = \mathbf{0}$

$\mathbf{x} \cdot \mathbf{y} = 0 \;\Leftrightarrow\; \mathbf{x}, \mathbf{y}$ orthogonal

Quadratische Matrizen

Einheitsmatrix $\mathbf{E} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$

Diagonalmatrix $\mathbf{D} = \mathrm{diag}\{\mathrm{d_i}\} = \begin{bmatrix} \mathrm{d_1} & & \\ & \ddots & \\ & & \mathrm{d_n} \end{bmatrix}$

Inverse Matrix $\mathbf{A}^{-1} = \frac{1}{\mathrm{det}\mathbf{A}}\mathrm{adj}\mathbf{A}$

$\mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{E}$

$(\mathbf{A} \cdot \mathbf{B})^{-1} = \mathbf{B}^{-1} \cdot \mathbf{A}^{-1}$

Symmetrische Matrix $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$

Schiefsymmetrische Matrix $\mathbf{A} = -\mathbf{A}^{\mathrm{T}}$

Zerlegung $\mathbf{A} = \underbrace{\frac{1}{2}(\mathbf{A} + \mathbf{A}^{\mathrm{T}})}_{\mathbf{B} = \mathbf{B}^{\mathrm{T}}} + \underbrace{\frac{1}{2}(\mathbf{A} - \mathbf{A}^{\mathrm{T}})}_{\mathbf{C} = -\mathbf{C}^{\mathrm{T}}}$

Schiefsymmetrische 3 × 3 Matrix $\tilde{\mathbf{a}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$

$$\tilde{\mathbf{a}} \cdot \mathbf{b} = \mathbf{a} \times \mathbf{b}$$
$$\tilde{\mathbf{a}} \cdot \mathbf{b} = -\tilde{\mathbf{b}} \cdot \mathbf{a} = \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$
$$\tilde{\mathbf{a}} \cdot \tilde{\mathbf{b}} = \mathbf{ba} - (\mathbf{a} \cdot \mathbf{b})\mathbf{E}$$
$$(\widetilde{\tilde{\mathbf{a}} \cdot \mathbf{b}}) = \mathbf{ba} - \mathbf{ab}$$

↕ Rösselsprung

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Symmetrische, positiv definite Matrix: $\mathbf{x} \cdot \mathbf{A} \cdot \mathbf{x} > 0 \quad \forall\, \mathbf{x} \neq \mathbf{0}$

⇔ Hauptabschnittsdeterminanten $H_\alpha > 0, \quad \alpha = 1(1)n$

⇔ Eigenwerte $\lambda_\alpha > 0, \quad \alpha = 1(1)n$

Symmetrische, positiv semidefinite Matrix: $\mathbf{x} \cdot \mathbf{A} \cdot \mathbf{x} \geq 0 \quad \forall\, \mathbf{x}$

⇔ Hauptabschnittsdeterminanten $H_\alpha \geq 0, \quad \alpha = 1(1)n$

⇔ Eigenwerte $\lambda_\alpha \geq 0, \quad \alpha = 1(1)n$

Orthogonale Matrix $\mathbf{A}^{-1} = \mathbf{A}^T,\ \mathbf{A} \cdot \mathbf{A}^T = \mathbf{A}^T \cdot \mathbf{A} = \mathbf{E}$

Determinante $\det \mathbf{A} = \sum_{i=1}^{n} A_{ik} B_{ik} = \sum_{k=1}^{n} A_{ik} B_{ik}$

Adjungierte Matrix $\operatorname{adj} \mathbf{A} = \begin{bmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & \ddots & \vdots \\ B_{n1} & \cdots & B_{nn} \end{bmatrix}^T$

Adjunkte eines Elementes einer Matrix

$$B_{ik} = (-1)^{i+k} \det \begin{bmatrix} A_{11} & \cdots & A_{1,k-1} & A_{1,k+1} & \cdots & A_{1n} \\ \vdots & \ddots & & & & \vdots \\ A_{i-1,1} & & & & & A_{i-1,n} \\ A_{i+1,1} & & & & & A_{i+1,n} \\ \vdots & & & & \ddots & \vdots \\ A_{n1} & \cdots & A_{n,k-1} & A_{n,k+1} & \cdots & A_{nn} \end{bmatrix}$$