Extracted from Doctoral Thesis Jörg Fehr [1].

Gramian Matrix Based Model Reduction

For first order systems the well known balanced truncation as explained e.g. in [2, 3] has a-priori error bounds [4] and asymptotic stability is preserved in the reduced order system. This combination of facts makes the method very attractive for automated and error controlled model reduction. For first order systems various balancing reduction methods exist, e.g. Lyapunov balancing or frequency weighted balancing. For model reduction based on frequency weighted Gramian matrices a frequency domain representation is beneficial. In [5] it is proven that if the matrix $\hat{\boldsymbol{A}}$ is Hurwitz and asymptotically stable, the Gramian matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ can be expressed by an integral expression in the frequency domain as

$$\boldsymbol{P} = \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\omega\boldsymbol{I} - \hat{\boldsymbol{A}})^{-1}\cdot\hat{\boldsymbol{B}}\cdot\hat{\boldsymbol{B}}^{T}\cdot(-i\omega\boldsymbol{I} - \hat{\boldsymbol{A}}^{T})^{-1}d\omega, \tag{1}$$

$$\boldsymbol{Q} = \frac{1}{2\pi}\int_{-\infty}^{\infty}(-i\omega\boldsymbol{I} - \hat{\boldsymbol{A}}^{T})^{-1}\cdot\hat{\boldsymbol{C}}^{T}\cdot\hat{\boldsymbol{C}}\cdot(i\omega\boldsymbol{I} - \hat{\boldsymbol{A}})^{-1}d\omega. \tag{2}$$

For model reduction issues an equivalent transformation into a balanced form is not necessary. It is adequate to retain only those states in the projection subspace span$(\boldsymbol{V}_f)$ which coincide with the biggest $2n$ HSV. The error of the reduced order system

$$\boldsymbol{H}_E(s) = \boldsymbol{H}(s) - \bar{\boldsymbol{H}}(s) \tag{3}$$

is then bounded by twice the sum of neglected HSVs

$$\|\boldsymbol{H}_E\|_{\mathcal{H}_\infty} \leq 2\sum_{i=2n+1}^{2N}\sigma_i. \tag{4}$$

A proof of the error bound is e.g. given in [3].

In the following, the balancing model reduction of second order systems is the center of attention. In this context, second order Gramian matrices are very important as far as model reduction of second order systems is concerned. However, contrary to first order systems, more than two Gramian matrices exist for a second order system.

A universally applicable error bound for all second order balancing reduction methods is not available. In addition, as explained in [6], even the stability of the reduced system cannot be guaranteed.

For the low rank ADI based algorithms the calculation of frequency weighted second order Gramian matrices is still an open topic. That is why here two other methods are used based on a two-step approach, introduced in [7]. In this case, the large scale model is reduced to a medium scale model in a first step, e.g. with an automated Krylov-subspace based model reduction technique. Subsequently, the Gramian matrices of the medium scale model can be calculated by an analytic solution of the model diagonalized with a congruence transformation. The other method is based on the fact that the matrix integral needed for calculating the Gramian matrices can be approximated by quadratures using integral kernel snapshots in combination with a POD based reduction. This method can be viewed as an extension of the Poor Man's Truncated Balanced Reduction (TBR) [8] scheme for second order systems.

Gramian Matrices for Second Order Systems

The Gramian matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ can be used to identify the most controllable and observable states of the linear time invariant first order state space system, compare [3, 2]. They measure how much energy is needed to control a state respectively how much energy is observable for a certain state. According to [9], second order Gramian matrices identify the important positions and velocities in the input / output (I/O) map of a second order system. For second order systems more than two Gramian matrices exist, see e.g. [6, 9]. The first order Gramian matrices of size $2N \times 2N$ are partitioned into

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}_p & * \\ * & \boldsymbol{P}_v \end{bmatrix}, \qquad \boldsymbol{Q} = \begin{bmatrix} \boldsymbol{Q}_p & * \\ * & \boldsymbol{Q}_v \end{bmatrix}, \tag{5}$$

where all blocks have the size $N \times N$ and only the diagonal elements are important to identify the most important positions and velocities, compare e.g. [6]. The position controllability Gramian matrix $\boldsymbol{P}_p$ identifies the most easily controllable positions and the position observability Gramian matrix $\boldsymbol{Q}_p$ defines the most easily observable positions. However, the velocity controllability Gramian matrix $\boldsymbol{P}_v$ and the velocity observability Gramian matrix $\boldsymbol{Q}_v$ tell how the I/O energy is distributed among the velocities.

In literature, different types of model reduction techniques based on second order Gramian matrices are proposed [9, 6, 10, 11]. Current research focuses on the question which one of the different reduction techniques is the most appropriate for model reduction of second order mechanical systems. They differ depending on which eigenvalues of the four system invariants are balanced simultaneously. Therefore, four different reduction methods are introduced and later compared. However, the implementation of these methods is a first try and further research is necessary to examine which are the most appropriate methods for different second order models. System invariants of a second order system are, as described in [6],

$$\text{the position singular values} \qquad \boldsymbol{\sigma}_p = \sqrt{\boldsymbol{\lambda}(\boldsymbol{P}_p \cdot \boldsymbol{Q}_p)}, \tag{6}$$

$$\text{the velocity singular values} \qquad \boldsymbol{\sigma}_v = \sqrt{\boldsymbol{\lambda}(\boldsymbol{P}_v \cdot \boldsymbol{M}_e^T \cdot \boldsymbol{Q}_v \cdot \boldsymbol{M}_e)}, \tag{7}$$

$$\text{the position velocity singular values} \qquad \boldsymbol{\sigma}_{pv} = \sqrt{\boldsymbol{\lambda}(\boldsymbol{P}_p \cdot \boldsymbol{M}_e^T \cdot \boldsymbol{Q}_v \cdot \boldsymbol{M}_e)}, \tag{8}$$

$$\text{the velocity position singular values} \qquad \boldsymbol{\sigma}_{vp} = \sqrt{\boldsymbol{\lambda}(\boldsymbol{P}_v \cdot \boldsymbol{Q}_p)}, \tag{9}$$

and are the square roots of the eigenvalues of the corresponding matrix. Additionally, the velocity singular values $\boldsymbol{\sigma}_v$ and the position velocity singular $\boldsymbol{\sigma}_{pv}$ values can be calculated by using the position velocity Gramian matrix $\boldsymbol{Q}_{pv}$. Therefore, Equations (7) and (8) are each expanded with two identity matrices $\boldsymbol{I} = \boldsymbol{M}_e^T \cdot \boldsymbol{M}_e^{-T} = \boldsymbol{M}_e^{-1} \cdot \boldsymbol{M}_e$ before and after the velocity observability matrix $\boldsymbol{Q}_v$

$$\begin{aligned} \boldsymbol{\sigma}_v &= \sqrt{\boldsymbol{\lambda}(\boldsymbol{P}_v \cdot \boldsymbol{M}_e^T \cdot \underbrace{\boldsymbol{M}_e^T \cdot \boldsymbol{M}_e^{-T}}_{\boldsymbol{I}} \cdot \boldsymbol{Q}_v \cdot \underbrace{\boldsymbol{M}_e^{-1} \cdot \boldsymbol{M}_e}_{\boldsymbol{I}} \cdot \boldsymbol{M}_e)} \\ &= \sqrt{\boldsymbol{\lambda}(\boldsymbol{P}_v \cdot \boldsymbol{M}_e^T \cdot \boldsymbol{M}_e^T \cdot \boldsymbol{Q}_{pv} \cdot \boldsymbol{M}_e \cdot \boldsymbol{M}_e)}, \end{aligned} \tag{10}$$

$$\begin{aligned} \boldsymbol{\sigma}_{pv} &= \sqrt{\boldsymbol{\lambda}(\boldsymbol{P}_p \cdot \boldsymbol{M}_e^T \cdot \underbrace{\boldsymbol{M}_e^T \cdot \boldsymbol{M}_e^{-T}}_{\boldsymbol{I}} \cdot \boldsymbol{Q}_v \cdot \underbrace{\boldsymbol{M}_e^{-1} \cdot \boldsymbol{M}_e}_{\boldsymbol{I}} \cdot \boldsymbol{M}_e)} \\ &= \sqrt{\boldsymbol{\lambda}(\boldsymbol{P}_p \cdot \boldsymbol{M}_e^T \cdot \boldsymbol{M}_e^T \cdot \boldsymbol{Q}_{pv} \cdot \boldsymbol{M}_e \cdot \boldsymbol{M}_e)} \end{aligned} \tag{11}$$

and are now expressed with $\boldsymbol{Q}_{pv}$. Based on the different singular values, different balanced realizations for second order systems can be defined. Here, the definitions from [6] are modified by using the position controllability Gramian matrix of the dual system $\boldsymbol{Q}_{pv}$ instead of the velocity observability Gramian matrix $\boldsymbol{Q}_v$ because for reduction with $\boldsymbol{Q}_{pv}$ in [11] an error estimator is available and in previous work reductions based on $\boldsymbol{Q}_{pv}$ achieved excellent results.

Frequency Weighted Gramian Matrices for Second Order Systems

Often, when examining mechanical systems, certain frequency ranges are of special interest. Frequency weighted balanced truncation for first order systems was introduced by [4] and an improved version is given, e.g., by [12]. Frequency ranges are emphasized by applying suitable frequency filters. The input is filtered by $\boldsymbol{W}_i(s)$ whereas the output is filtered by $\boldsymbol{W}_o(s)$. Such frequency weighted Gramian matrices tell what are the important states in a certain frequency range. In general terms, the frequency share of the excitations outside the interesting frequency range is removed by the input filter and the frequency share of the outputs outside the frequency range are also removed. Frequency weighted Gramian matrices are calculated by substituting the input to state map $\boldsymbol{L}^{-1}(s) \cdot \boldsymbol{B}_e$ with a suitable weighted variant of the map $\boldsymbol{L}^{-1}(s) \cdot \boldsymbol{B}_e \cdot \boldsymbol{W}_i(s)$ and additionally weight the output $\boldsymbol{C}_e$ with $\boldsymbol{W}_o(s)$. The frequency weighted version of the second order Gramian matrices are then obtained. They read

$$\boldsymbol{P}_p^i = \frac{1}{2\pi}\int_{-\infty}^{\infty} \boldsymbol{L}^{-1}(i\omega)\cdot(\boldsymbol{B}_e)\cdot\boldsymbol{W}_i(i\omega)\cdot\boldsymbol{W}_i^H(i\omega)\cdot(\boldsymbol{B}_e^T)\cdot\boldsymbol{L}^{-H}(i\omega)d\omega, \tag{12}$$

$$\boldsymbol{P}_v^i = \frac{1}{2\pi}\int_{-\infty}^{\infty} \boldsymbol{L}^{-1}(i\omega)\cdot(i\omega\boldsymbol{B}_e)\cdot\boldsymbol{W}_i(i\omega)\cdot\boldsymbol{W}_i^H(i\omega)\cdot(-i\omega\boldsymbol{B}_e^T)\cdot\boldsymbol{L}^{-H}(i\omega)d\omega, \tag{13}$$

$$\begin{aligned}\boldsymbol{Q}_p^i = \frac{1}{2\pi}\int_{-\infty}^{\infty} & (-i\omega\boldsymbol{M}_e^T+\boldsymbol{D}_e^T)\cdot\boldsymbol{L}^{-H}(i\omega)\cdot\boldsymbol{C}_e^T\cdot\boldsymbol{W}_o^H(i\omega)\\ &\cdot\boldsymbol{W}_o(i\omega)\cdot\boldsymbol{C}_e\cdot\boldsymbol{L}^{-1}(i\omega)\cdot(i\omega\boldsymbol{M}_e+\boldsymbol{D}_e)d\omega,\end{aligned} \tag{14}$$

$$\boldsymbol{Q}_v^i = \frac{1}{2\pi}\int_{-\infty}^{\infty} \boldsymbol{M}_e^T\cdot\boldsymbol{L}^{-H}(i\omega)\cdot\boldsymbol{C}_e^T\cdot\boldsymbol{W}_o^H(i\omega)\cdot\boldsymbol{W}_o(i\omega)\cdot\boldsymbol{C}_e\cdot\boldsymbol{L}^{-1}(i\omega)\cdot\boldsymbol{M}_e^{-1}d\omega, \tag{15}$$

$$\boldsymbol{Q}_{pv}^i = \frac{1}{2\pi}\int_{-\infty}^{\infty} \boldsymbol{L}^{-H}(i\omega)\cdot\boldsymbol{C}_e^T\cdot\boldsymbol{W}_o^H(i\omega)\cdot\boldsymbol{W}_o(i\omega)\cdot\boldsymbol{C}_e\cdot\boldsymbol{L}^{-1}(i\omega)d\omega, \tag{16}$$

where $\boldsymbol{L}^H$ and $\boldsymbol{W}^H$ are the conjugated transposes of the complex matrices $\boldsymbol{L}$ and $\boldsymbol{W}$. For mechanical systems, the filter matrices are often not available directly but the interesting frequency range is known. In this case, the input and output filter matrices $\boldsymbol{W}_i(s)$ and $\boldsymbol{W}_o(s)$ are the transfer functions of ideal band pass filters, see [3]. The frequency weighted version of the second order position controllability Gramian matrix then reads

$$\boldsymbol{P}_p^i = \frac{1}{2\pi}\int_{-\omega_2}^{-\omega_1} \boldsymbol{L}^{-1}(i\omega)\cdot\boldsymbol{B}_e\cdot\boldsymbol{B}_e^T\cdot\boldsymbol{L}^{-H}(i\omega)d\omega + \frac{1}{2\pi}\int_{\omega_2}^{\omega_1} \boldsymbol{L}^{-1}(i\omega)\cdot\boldsymbol{B}_e\cdot\boldsymbol{B}_e^T\cdot\boldsymbol{L}^{-H}(i\omega)d\omega. \tag{17}$$

The integration variable in the first summand is substituted by $\bar{\omega} = -\omega$ and afterwards the integration boundaries are switched

$$\begin{aligned}\boldsymbol{P}_p^i &= \frac{1}{2\pi}\int_{\omega_2}^{\omega_1} \boldsymbol{L}^{-1}(i\bar{\omega})\cdot\boldsymbol{B}_e\cdot\boldsymbol{B}_e^T\cdot\boldsymbol{L}^{-H}(i\bar{\omega})(-1)d\bar{\omega} + \frac{1}{2\pi}\int_{\omega_1}^{\omega_2} \boldsymbol{L}^{-1}(i\bar{\omega})\cdot\boldsymbol{B}_e\cdot\boldsymbol{B}_e^T\cdot\boldsymbol{L}^{-H}(i\omega)d\omega\\ &= \frac{1}{2\pi}\int_{\omega_1}^{\omega_2} \boldsymbol{L}^{-1}(i\bar{\omega})\cdot\boldsymbol{B}_e\cdot\boldsymbol{B}_e^T\cdot\boldsymbol{L}^{-H}(i\bar{\omega})d\bar{\omega} + \frac{1}{2\pi}\int_{\omega_1}^{\omega_2} \boldsymbol{L}^{-1}(i\omega)\cdot\boldsymbol{B}_e\cdot\boldsymbol{B}_e^T\cdot\boldsymbol{L}^{-H}(i\omega)d\omega.\end{aligned} \tag{18}$$

Another representation of the Gramian matrices weighted with ideal band pass filters is

$$\begin{aligned} \boldsymbol{P}_p^i(\omega_1,\omega_2) &= \boldsymbol{P}_p^i(\omega_2) - \boldsymbol{P}_p^i(\omega_1), & \boldsymbol{P}_v^i(\omega_1,\omega_2) &= \boldsymbol{P}_v^i(\omega_2) - \boldsymbol{P}_v^i(\omega_1), \\ \boldsymbol{Q}_p^i(\omega_1,\omega_2) &= \boldsymbol{Q}_p^i(\omega_2) - \boldsymbol{Q}_p^i(\omega_1), & \boldsymbol{Q}_v^i(\omega_1,\omega_2) &= \boldsymbol{Q}_v^i(\omega_2) - \boldsymbol{Q}_v^i(\omega_1), \\ \boldsymbol{Q}_{pv}^i(\omega_1,\omega_2) &= \boldsymbol{Q}_{pv}^i(\omega_2) - \boldsymbol{Q}_{pv}^i(\omega_1), \end{aligned} \tag{19}$$

with

$$\boldsymbol{P}_p^i(\omega_1) = \frac{1}{2\pi}\int_{-\omega_1}^{\omega_1} \boldsymbol{L}^{-1}(i\omega)\cdot\boldsymbol{B}_e\cdot\boldsymbol{B}_e^T\cdot\boldsymbol{L}^{-H}(i\omega)d\omega, \tag{20}$$

$$\boldsymbol{P}_v^i(\omega_1) = \frac{1}{2\pi}\int_{-\omega_1}^{\omega_1} \boldsymbol{L}^{-1}(i\omega)\cdot(i\omega\boldsymbol{B}_e)\cdot(-i\omega\boldsymbol{B}_e^T)\cdot\boldsymbol{L}^{-H}(i\omega)d\omega, \tag{21}$$

$$\boldsymbol{Q}_p^i(\omega_1) = \frac{1}{2\pi}\int_{-\omega_1}^{\omega_1} (-i\omega\boldsymbol{M}_e^T + \boldsymbol{D}_e^T)\cdot\boldsymbol{L}^{-H}(i\omega)\cdot\boldsymbol{C}_e^T\cdot\boldsymbol{C}_e\cdot\boldsymbol{L}^{-1}(i\omega)\cdot(i\omega\boldsymbol{M}_e + \boldsymbol{D}_e)d\omega, \tag{22}$$

$$\boldsymbol{Q}_v^i(\omega_1) = \frac{1}{2\pi}\int_{-\omega_1}^{\omega_1} \boldsymbol{M}_e^T\cdot\boldsymbol{L}^{-H}(i\omega)\cdot\boldsymbol{C}_e^T\cdot\boldsymbol{C}_e\cdot\boldsymbol{L}^{-1}(i\omega)\cdot\boldsymbol{M}_e^{-1}d\omega, \tag{23}$$

$$\boldsymbol{Q}_{pv}^i(\omega_1) = \frac{1}{2\pi}\int_{-\omega_1}^{\omega_1} \boldsymbol{L}^{-H}(-i\omega)\cdot\boldsymbol{C}_e^T\cdot\boldsymbol{C}_e\cdot\boldsymbol{L}^{-1}(-i\omega)d\omega. \tag{24}$$

The decisive advantage of the usage of frequency weighted Gramian matrices can be seen in Figures 1 and 2. Two models are compared. One model is reduced with the $n = 24$ dominant eigenvectors of the Gramian position matrix $\boldsymbol{P}_p$, the other one is reduced with the $n = 24$ dominant eigenvectors of the frequency weighted Gramian matrix $\boldsymbol{P}_p^i$, where the interesting frequency range $[f_{min}, f_{max}] = [1, 20\text{ Hz}]$ was chosen optimally with respect to the later harmonic excitation $\boldsymbol{F}_{harm}^{rack}$. Furthermore, the results of the two models are compared with the translation of the nonlinear finite element model. The accuracy of the reduced model which was optimized with a frequency weighted Gramian matrix is very good. The model is optimized for this excitation frequency. On the contrary, the reduction with an unweighted Gramian matrix $\boldsymbol{P}_p$ leads to wrong results. Based on the convincing results derived from the usage of frequency weighted Gramian matrices, in the following the model reduction procedures exclusively use frequency weighted Gramian matrices $\boldsymbol{P}_\alpha^i$ and $\boldsymbol{Q}_\beta^i$ instead of unweighted Gramian matrices $\boldsymbol{P}_\alpha$ and $\boldsymbol{Q}_\alpha$ with $\alpha \in \{p, v\}$ and $\beta \in \{p, pv\}$.

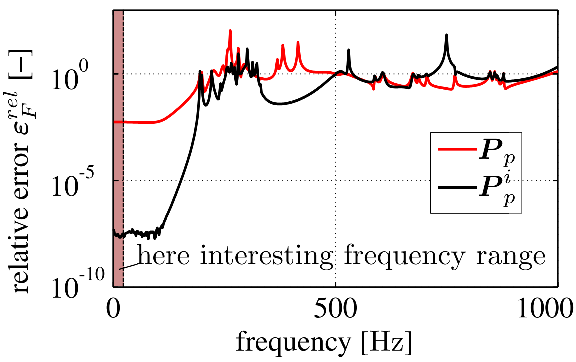

Figure 1: Influence of frequency weighting on the relative error ε_F^{rel} of the rack reduced to the same reduction size $n = 24$.

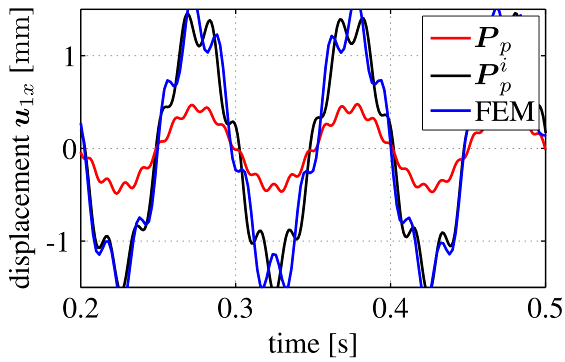

Figure 2: Influence of frequency weighting on the time response of the harmonically excited rack

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